ANICK'S FIBRATION AND THE ODD PRIMARY HOMOTOPY EXPONENT OF SPHERES

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ABSTRACT. For primes $p \geq 3$, Cohen, Moore, and Neisendorfer showed that the exponent of the p-torsion in the homotopy groups of S^{2n+1} is p^n. This was obtained as a consequence of a thorough analysis of the homotopy theory of Moore spaces. Anick further developed this for $p \geq 5$ by constructing a homotopy fibration $S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}$ whose connecting map is degree p^r on the bottom cell. A much simpler construction of such a fibration for $p \geq 3$ was given by Gray and the author using new methods. In this paper the new methods are used to start over, first constructing Anick's fibration for $p \geq 3$, and then using it to obtain the exponent result for spheres.

1. INTRODUCTION

Cohen, Moore, and Neisendorfer's [CMN1, CMN2, N2] determination of the odd primary homotopy exponent of spheres is a miletone in homotopy theory. They showed that if p is an odd prime then p^n is the least power of p which annihilates the p-torsion in $\pi_*(S^{2n+1})$. This was obtained as a consequence of a geometric result. Localize spaces and maps at p. Let $E^2 \colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ be the double suspension. For each $n \geq 1$, they constructed a map $\varphi \colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ with the property that the composite $\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$ is homotopic to the pth-power map. Thus $p \cdot \pi_*(S^{2n+1})$ factors through $\pi_*(S^{2n-1})$. Inducting on n therefore shows that $p^n \cdot \pi_m(S^{2n+1}) = 0$ for all $m > 2n + 1$.

The map φ was obtained as a consequence of very thorough analysis of the homotopy theory of Moore spaces. The mod-p^r Moore space $P^m(p^r)$ is the homotopy cofiber of the degree p^r map on S^{m-1}. Let F be the homotopy fiber of the pinch map $q \colon P^m(p^r) \longrightarrow S^m$ onto the top cell. When $m = 2n + 1$, Cohen, Moore, and Neisendorfer studied the homotopy fibration sequence

$$\Omega^2 S^{2n+1} \longrightarrow \Omega F \longrightarrow \Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}$$

and painstakingly determined a decomposition of ΩF. In particular, one of the factors of ΩF is S^{2n-1}, giving a composite $\varphi_r \colon \Omega^2 S^{2n+1} \longrightarrow \Omega F^{2n+1} \longrightarrow S^{2n-1}$. The $r = 1$ case is the map φ above.

It was later shown that the map φ_r fits in a homotopy fibration sequence

$$(1) \qquad \Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}.$$

This was proved by Anick [A] for $p \geq 5$ by using Cohen, Moore, and Neisendorfer's work as the base-case in a complex induction. A much simpler proof was given in [GT] for $p \geq 3$ by using new methods.

These methods raised the possibility of producing a different proof of the exponent result for spheres which depends more on properties of spheres than Moore spaces. The purpose of this paper is to give such a proof. The idea is to first construct a homotopy fibration as in (1) and then obtain the exponent result as a consequence. Explicitly, from first principles, we prove the following.

Theorem 1.1. *Let $p \geq 3$ and $r \geq 1$. There is a homotopy fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1}$$

with the property that the composition $\Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$ is homotopic to the p^r-power map.

Consequently, inducting on the $r = 1$ case gives the exponent result.

Corollary 1.2. *The p-torsion in $\pi_*(S^{2n+1})$ is annihilated by p^n.* □

One advantage of our work compared to Cohen, Moore, and Neisendorfer's is that it bypasses all the hard work in [CMN1] to find and then keep track of higher order torsion in the homotopy groups of Moore spaces. A second advantage is that it works simultaneously for all odd primes, whereas the $p = 3$ case in [N1] required technical modifications to the $p \geq 5$ cases in [CMN1, CMN2]. A third advantage is that it constructs Anick's fibration at the same time as proving the exponent result. On the other hand, by intent, our work says little about the homotopy theory of Moore spaces whereas [CMN1] gives a very thorough account of it.

This work is partly motivated for two other reasons as well. The first concerns the homotopy fiber W_n of the double suspension. A long-standing conjecture is that W_n is a double loop space. In particular, potentially $W_n \simeq \Omega^2 T^{2np+1}(p)$. Such a homotopy equivalence would have deep implications, one of which being a determination of many differentials in the E_2-term of the EHP-spectral sequence calculating the homotopy groups of spheres. The sticking point is that W_n naturally arises from an EHP-perspective while $T^{2np+1}(p)$ naturally arises from a Cohen, Moore, Neisendorfer (CMN) perspective, and it is not known how to reconcile the two. Our starting point in constructing the space $T^{2n+1}(p^r)$ is to use Gray's [G] construction of a classifying space BW_n of W_n. So this paper can be thought of as an attempt to fuse the CMN and EHP-perspectives, in the hope that it will tell us more about whether $W_n \simeq \Omega^2 T^{2np+1}(p)$. The second motivation concerns whether there are analogues of Anick's fibration at the prime 2. For Hopf invariant one reasons the space

$T^{2n+1}(2)$ cannot exist if $n \notin \{2, 4, 8\}$, but Cohen has conjectured that $T^{2n+1}(2^r)$ does exist if $r > 1$. The approach in [GT] would come close to constructing such spaces if it were known that BW_n is a homotopy associative H-space when localized at 2. However, this is not known and is probably not the case. The approach to constructing the space $T^{2n+1}(p^r)$ here when p is odd is a variant of that in [GT], which bypasses the issue of whether BW_n is an H-space, and so raises the possibility of positively answering Cohen's conjecture. This will be pursued in later work.

2. An outline of the proof of Theorem 1.1

In this section we prove Theorem 1.1, deferring details to later sections in order to make the thrust of the proof clear. We begin by defining some spaces and maps, and stating their relevant properties.

Let p be an odd prime and $r \in \mathbb{N}$. Let $p^r \colon S^{2n+1} \longrightarrow S^{2n+1}$ be the map of degree p^r. Let $S^{2n+1}\{p^r\}$ be its homotopy fiber. As p is an odd prime, S^{2n+1} is an H-space, its p^r-power map is homotopic to the degree p^r map, and the p^r-power map is an H-map. In particular, $S^{2n+1}\{p^r\}$ is an H-space. In [N3] it was shown that the p^r-power map on $S^{2n+1}\{p^r\}$ is null homotopic.

In [G] it was shown that the fiber W_n of the double suspension has a classifying space BW_n and there are homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$$

with the property that the composite $j \circ \nu$ is homotopic to ΩH, where H is the pth-James-Hopf invariant. Further, BW_n is an H-space and the maps ν and j are H-maps (the $p = 3$ case of this being proved in [T2]). Based on the fact that ΩH has order p, it was shown in [G] that the composite $BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{p} \Omega^2 S^{2np+1}$ is null homotopic, implying that j lifts to a map $\overline{S} \colon BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\}$. In [T1], based on a result in [S], it was shown that $\overline{S}$ can be chosen to be an H-map. In Lemma 3.2 we will show that the homotopy fiber of $\overline{S}$ is ΩW_{np}. Let S be the composite

$$S \colon \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{\overline{S}} \Omega^2 S^{2np+1}\{p\}.$$

Note that S is an H-map as it the composite of H-maps.

Now we turn to setting up the proof of Theorem 1.1. Define the space Y and the maps f, g, and h by the homotopy pullback diagram

$$
(2) \qquad
\begin{array}{ccccc}
S^{2n-1} & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & \Omega W_{np} \\
\Big\| & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow \\
S^{2n-1} & \xrightarrow{\ E^2\ } & \Omega^2 S^{2n+1} & \xrightarrow{\ \nu\ } & BW_n \\
& & \Big\downarrow{\scriptstyle S} & & \Big\downarrow{\scriptstyle \overline{S}} \\
& & \Omega^2 S^{2np+1}\{p\} & =\!=\!= & \Omega^2 S^{2np+1}\{p\}.
\end{array}
$$

As it stands, Cohen, Moore, and Neisendorfer's work tells us that there is a lift

$$
\begin{array}{ccc}
 & & \Omega^2 S^{2n+1} \\
 & \nearrow & \Big\downarrow{\scriptstyle p^r} \\
S^{2n-1} & \xrightarrow{\ E^2\ } & \Omega^2 S^{2n+1}
\end{array}
$$

for each $r \geq 1$. To reproduce this in our case, we will find successive lifts

$$
(3) \qquad
\begin{array}{ccc}
 & & \Omega^2 S^{2n+1} \\
 & \overset{\ell_2}{\diagup} \quad \overset{\ell_1}{\diagup} & \Big\downarrow{\scriptstyle p^r} \\
S^{2n-1} & \xrightarrow{\ f\ } \quad Y \xrightarrow{\ h\ } & \Omega^2 S^{2n+1}.
\end{array}
$$

Both ℓ_1 and ℓ_2 are constructed as consequences of certain extensions. To describe how ℓ_1 comes about, consider the pinch map $P^{2n+1}(p^r) \xrightarrow{\ q\ } S^{2n+1}$ onto the top cell. This factors as the composite $P^{2n+1}(p^r) \xrightarrow{\ i\ } S^{2n+1}\{p^r\} \longrightarrow S^{2n+1}$ where i is the inclusion of the bottom Moore space. The factorization determines an extended homotopy pullback diagram

$$
(4) \qquad
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\ \partial_E\ } & E & \longrightarrow & F & \longrightarrow & \Omega S^{2n+1} \\
\Big\downarrow & & \Big\| & & \Big\downarrow & & \Big\downarrow \\
\Omega S^{2n+1}\{p^r\} & \longrightarrow & E & \longrightarrow & P^{2n+1}(p^r) & \xrightarrow{\ i\ } & S^{2n+1}\{p^r\} \\
 & & & & \Big\downarrow{\scriptstyle q} & & \Big\downarrow \\
 & & & & S^{2n+1} & =\!=\!= & S^{2n+1}
\end{array}
$$

which defines the spaces E and F and the map ∂_E.

Proposition 2.1. *There is an extension*

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\ \partial_E\ } & E \\
\Big\downarrow{\scriptstyle S} & \diagup{\scriptstyle e_1} & \\
\Omega^2 S^{2np+1}\{p\} & &
\end{array}
$$

for some map e_1.

By (4) the map ∂_E factors through $\Omega S^{2n+1}\{p^r\}$. Using this, let e_1' be the composite

$$e_1' \colon \Omega S^{2n+1}\{p^r\} \longrightarrow E \xrightarrow{\ e_1\ } \Omega^2 S^{2np+1}\{p\}.$$

Then Proposition 2.1 implies that there is a homotopy commutative square

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{p^r\} \\
\big\downarrow{\scriptstyle S} & & \big\downarrow{\scriptstyle e_1'} \\
\Omega^2 S^{2np+1}\{p\} & =\!\!=\!\!= & \Omega^2 S^{2np+1}\{p\}.
\end{array}
$$

This square determines an extended homotopy pullback diagram

(5)
$$
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\ \ell_1\ } & Y & \longrightarrow & X & \xrightarrow{\ q_X\ } & \Omega S^{2n+1} \\
\big\| & & \big\downarrow{\scriptstyle h} & & \big\downarrow & & \big\| \\
\Omega^2 S^{2n+1} & \xrightarrow{\ p^r\ } & \Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \longrightarrow & \Omega S^{2n+1} \\
& & \big\downarrow{\scriptstyle S} & & \big\downarrow{\scriptstyle e_1'} & & \\
& & \Omega^2 S^{2np+1}\{p\} & =\!\!=\!\!= & \Omega^2 S^{2np+1}\{p\} & &
\end{array}
$$

which defines the space X and the maps q_X and ℓ_1. In particular, ℓ_1 is a choice of a lift of the p^r-power map with the additional property that it is a connecting map of a homotopy fibration.

Further, the factorization of e_1' through E determines a homotopy pullback diagram

(6)
$$
\begin{array}{ccccc}
\Omega P^{2n+1}(p^r) & \xrightarrow{\ i_X\ } & X & \longrightarrow & R \\
\big\| & & \big\downarrow & & \big\downarrow \\
\Omega P^{2n+1}(p^r) & \xrightarrow{\ \Omega i\ } & \Omega S^{2n+1}\{p^r\} & \longrightarrow & E \\
& & \big\downarrow{\scriptstyle e_1'} & & \big\downarrow{\scriptstyle e_1} \\
& & \Omega^2 S^{2np+1}\{p\} & =\!\!=\!\!= & \Omega^2 S^{2np+1}\{p\}
\end{array}
$$

which defines the space R and the map i_X. Note that (5) and (6) imply that the composite $\Omega P^{2n+1}(p^r) \xrightarrow{\ i_X\ } X \xrightarrow{\ q_X\ } \Omega S^{2n+1}$ is homotopic to Ωq. The map i_X and the homotopy $\Omega q \simeq q_X \circ i_X$ will play an important role in proving the existence of the second extension.

Consider the homotopy fibration $Y \longrightarrow X \xrightarrow{\ q_X\ } \Omega S^{2n+1}$ and the map $Y \xrightarrow{\ g\ } \Omega W_{np}$ from (2).

Proposition 2.2. *There is an extension*

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\big\downarrow{\scriptstyle g} & \nearrow{\scriptstyle e_2} & \\
\Omega W_{np} & &
\end{array}
$$

for some map e_2.

Proposition 2.2 implies that there is an extended homotopy pullback diagram

(7)

$$
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\ell_2} & S^{2n-1} & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1} \\
\Big\| & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow & & \Big\| \\
\Omega^2 S^{2n+1} & \xrightarrow{\ell_1} & Y & \longrightarrow & X & \xrightarrow{q_X} & \Omega S^{2n+1} \\
& & \Big\downarrow{\scriptstyle g} & & \Big\downarrow & & \\
& & \Omega W_{np} & =\!\!=\!\!= & \Omega W_{np} & &
\end{array}
$$

which defines the space $T^{2n+1}(p^r)$ and the map ℓ_2.

Proof of Theorem 1.1: The top row in (7) is the asserted homotopy fibration. Relabel ℓ_2 as φ_r. All that is left to check is that φ_r has the property that $E^2 \circ \varphi_r$ is homotopic to the p^r-power map on $\Omega^2 S^{2n+1}$. This follows from juxtaposing the squares of connecting maps in (5) and (7), giving a homotopy commutative diagram

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\varphi_r} & S^{2n-1} \\
\Big\| & & \Big\downarrow{\scriptstyle f} \\
\Omega^2 S^{2n+1} & \xrightarrow{\ell_1} & Y \\
\Big\| & & \Big\downarrow{\scriptstyle h} \\
\Omega^2 S^{2n+1} & \xrightarrow{p^r} & \Omega^2 S^{2n+1},
\end{array}
$$

and noting that by (2) the composite $h \circ f$ is homotopic to E^2. $\square$

We close this section with some remarks regarding the proof of Theorem 1.1. The key is establishing the existence of the extensions in Propositions 2.1 and 2.2. These are obtained by filtering E and X respectively by certain homotopy pushouts. Inductively, extensions are produced as pushout maps. To determine that a pushout map exists, we use a slight generalization of a theorem in [GT]. One of the hypotheses of this theorem involves a certain map in the pushout being divisible by p^r, or having its suspension divisible by p^r. This is played off against the fact that the pth-power maps on $\Omega^2 S^{2np+1}\{p\}$ and ΩW_{np} are null homotopic in order to annihilate the obstructions to the existence of a pushout map.

It has already been pointed out that $S^{2np+1}\{p\}$ has a null homotopic p^{th}-power map, implying the same for $\Omega^2 S^{2np+1}\{p\}$. The case of ΩW_{np} needs to be stated clearly. In [CMN1] it was shown that the pth-power map on W_n is null homotopic, but this used the fact that the pth-power map on $\Omega^2 S^{2n+1}$ factors as the composite $\Omega^2 S^{2n+1} \xrightarrow{\varphi} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$. In [T2] the same factorization was used to show the stronger result that the pth-power map on BW_n is null homotopic. As we are trying to give a new proof of this factorization, we have to suppress knowledge of these exponent results. However, as will be seen in Section 3, the method of proof in [T2] can be applied to an alternate factorization of the pth-power map on $\Omega^2 S^{2np+1}$ as the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi}$

$S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$, which is independent of [CMN1], in order to show that the pth-power map on BW_{np} is null homotopic.

The remainder of the paper is organized as follows. Section 3 gives some preliminary background results related to BW_n. In Section 4 the generalized version of the extension theorem in [GT] is given. The p^r-divisibility condition is established in two contexts in Sections 5 and 6, corresponding to the contexts of Propositions 2.1 and 2.2. The two propositions are then proved in Section 7. Finally, in Section 8 we give some concluding remarks and consequences.

3. Background results

In this section we establish some background material related to BW_n. First, consider the map $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$ whose homotopy fiber is BW_n. In [T1] it was shown that the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ is homotopic to the pth-power map. The following lemma is an adaptation of [T2, 1.2].

Lemma 3.1. *The pth-power map on BW_{np} is null homotopic.*

Proof. Since BW_{np} is an H-space, it suffices to show that $\Sigma BW_{np} \xrightarrow{\Sigma p} \Sigma BW_{np}$ is null homotopic. As the pth-power map on $\Omega^2 S^{2np+1}$ factors as $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$, the composite $\nu \circ p$ factors as $\nu \circ E^2 \circ \phi$. Since ν and E^2 are consecutive maps in a homotopy fibration, their composite is null homotopic and so $\nu \circ p$ is null homotopic. Since ν is an H-map, $p \circ \nu \simeq \nu \circ p$, and so $p \circ \nu$ is null homotopic. Let C be the homotopy cofiber of ν. Then the null homotopy for $p \circ \nu$ implies that there is an extension

$$
\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_n & \xrightarrow{t} & C \\
& & \downarrow{\scriptstyle p} & \nearrow{\scriptstyle \lambda} & \\
& & BW_n & &
\end{array}
$$

for some map λ. By [G], the map $\Sigma^2 \nu$ has a right homotopy inverse. This implies that there is a homotopy decomposition $\Sigma^2 \Omega^2 S^{2n+1} \simeq \Sigma C \vee \Sigma^2 BW_n$. Thus Σt is null homotopic and so Σp is null homotopic. $\qquad\square$

Next, consider the H-map $BW_n \xrightarrow{\overline{S}} \Omega^2 S^{2np+1}\{p\}$ which lifts $BW_n \xrightarrow{j} \Omega^2 S^{2np+1}$. In Section 2 it was stated that the homotopy fiber of $\overline{S}$ is ΩW_{np}. We prove this now.

Lemma 3.2. *There is a homotopy fibration $\Omega W_{np} \longrightarrow BW_n \xrightarrow{\overline{S}} \Omega^2 S^{2np+1}\{p\}$.*

Proof. Consider the homotopy pullback diagram

$$
\begin{array}{ccccc}
M & \longrightarrow & \Omega S^{2np-1} & \xrightarrow{\ s\ } & \Omega^3 S^{2np+1} \\
\| & & \downarrow & & \downarrow \\
M & \longrightarrow & BW_n & \xrightarrow{\ \overline{S}\ } & \Omega^2 S^{2np+1}\{p\} \\
& & \downarrow{\scriptstyle j} & & \downarrow \\
& & \Omega^2 S^{2np+1} & = & \Omega^2 S^{2np+1}
\end{array}
$$

which defines the space M and the map s. Since this is a pullback of H-spaces and H-maps, the induced map s is an H-map. The James construction [J] implies that any H-map $t\colon \Omega\Sigma X \longrightarrow Z$, where Z is homotopy associative, is determined uniquely, up to homotopy, by $t \circ E$. In our case, $s \circ E$ is of degree 1 in homology and so $s \circ E \simeq E^3$. On the other hand, $\Omega S^{2np-1} \xrightarrow{\Omega E} \Omega^3 S^{2np+1}$ is another H-map with the property that $\Omega E \circ E \simeq E^3$. Thus $s \simeq \Omega E$. Hence $M \simeq \Omega W_{np}$. $\qquad\square$

Finally, we need to show the existence of a certain homotopy commutative diagram describing a property of the H-map $\Omega^2 S^{2n+1} \xrightarrow{\ S\ } \Omega^2 S^{2np+1}\{p\}$. In general, if Z is an H-space and there are maps $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$, let $f \cdot g$ be the product of f and g, obtained as the composite

$$
f \cdot g \colon X \times Y \xrightarrow{f \times g} Z \times Z \xrightarrow{\ m\ } Z
$$

where m is the multiplication on Z.

Lemma 3.3. *There is a homotopy commutative diagram*

$$
\begin{array}{ccc}
S^{2n-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\ E^2 \cdot 1\ } & \Omega^2 S^{2n+1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle S} \\
\Omega^2 S^{2n+1} & \xrightarrow{\ S\ } & \Omega^2 S^{2n+1}\{p\}
\end{array}
$$

where π_2 is the projection.

Proof. Observe that the composite $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\ S\ } \Omega^2 S^{2np+1}\{p\}$ is null homotopic by connectivity. The lemma now follows since S is an H-map. $\qquad\square$

4. AN EXTENSION THEOREM

Let $A \longrightarrow E' \longrightarrow E$ be a homotopy cofibration and suppose there is a map $F \longrightarrow E$. Define spaces F', Q, and X by the diagram of iterated homotopy pullbacks

$$
(8)\qquad
\begin{array}{ccccc}
X & = & X & = & X \\
\downarrow & & \downarrow & & \downarrow \\
Q & \longrightarrow & F' & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & E' & \longrightarrow & E.
\end{array}
$$

The following Lemma can be found in [G].

Lemma 4.1. *There is a homotopy equivalence $Q \xrightarrow{\simeq} A \times X$ such that the map $Q \longrightarrow A$ becomes the projection, and there is a homotopy pushout*

$$
\begin{array}{ccc}
Q \simeq A \times X & \longrightarrow & F' \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
X & \longrightarrow & F.
\end{array}
$$

Remark 4.2. Note that there may be many choices of the homotopy equivalence for Q, and therefore many possible homotopy classes for the map $A \times X \longrightarrow F'$. One situation where additional control can be imposed is in the case of a principal fibration. Suppose $X = \Omega X'$ and there is a homotopy fibration sequence $\Omega X' \longrightarrow F' \longrightarrow E' \longrightarrow X'$. Then there is a homotopy action $\theta \colon F' \times \Omega X' \longrightarrow F'$. As observed in [GT], the decomposition $Q \simeq A \times X$ can then be chosen so that the map $A \times X \longrightarrow F'$ is homotopic to the composite $A \times X \xrightarrow{a \times 1} F' \times X \xrightarrow{\theta} F'$, where a is any choice of a lift of $A \longrightarrow E'$ to F'. Note that such a lift exists by the decomposition of Q, and there may be many choices of a.

Theorem 4.3, a slight generalization of [GT, 2.3], gives conditions for when the homotopy pushout in Lemma 4.1 can be used to extend a map $F' \longrightarrow Z$ to a map $F \longrightarrow Z$. As notation, if A is a co-H space, let $\underline{p^r} \colon A \longrightarrow A$ be the map of degree p^r.

Theorem 4.3. *Given a diagram as in (8) in which A is a suspension. Suppose the homotopy equivalence $Q \simeq A \times X$ in Lemma 4.1 can be chosen so that the map $Q \simeq A \times X \longrightarrow F'$ is homotopic to a composite*

$$
A \times X \xrightarrow{a \times 1} M \times X \longrightarrow F'
$$

for some space M, where a has the property that $\Sigma a \simeq t \circ \underline{p^r}$ for some map t. Let Z be a homotopy associative H-space whose p^r-power map is null homotopic and suppose there is a map $F' \longrightarrow Z$. Then there is an extension

$$
\begin{array}{ccc}
F' & \longrightarrow & Z \\
\downarrow & & \| \\
F & \longrightarrow & Z.
\end{array}
$$

Proof. The theorem was proved in [GT, 2.3] in the special case where $X = \Omega X'$ and there is a homotopy fibration sequence $\Omega X' \longrightarrow F' \xrightarrow{f'} E' \longrightarrow X'$. By Remark 4.2, the decomposition $Q \simeq A \times X$ may be chosen so that the map $Q \simeq A \times X \longrightarrow F'$ is homotopic to the composite

$$
\overline{\theta} \colon A \times X \xrightarrow{a \times 1} F' \times X \longrightarrow F',
$$

where a is some lift of $A \longrightarrow E$. So in this case, $M = F'$. The existence of the extension is proved in [GT, 2.6] given the composite $\overline{\theta}$, the fact that Σa is divisible by p^r, and the H-space properties of Z.

In the more general case, the proof of [GT, 2.6] holds without change once $\overline{\theta}$ is replaced by the given composite $A \times X \xrightarrow{a \times 1} M \times X \longrightarrow F'$. $\hfill\square$

5. p^r-DIVISIBILITY I

In the next two sections we prove p^r-divisibility results which will eventually be used as input into Theorem 4.3. Assume homology is taken with mod-p coefficients.

The first p^r-divisibility result concerns the spaces E and F in the homotopy pullback diagram

$$
(9) \qquad
\begin{array}{ccccc}
E & \longrightarrow & F & \longrightarrow & \Omega S^{2n+1} \\
\| & & \downarrow & & \downarrow \\
E & \longrightarrow & P^{2n+1}(p^r) & \xrightarrow{\ i\ } & S^{2n+1}\{p^r\} \\
& & \downarrow{\scriptstyle q} & & \downarrow \\
& & S^{2n+1} & = & S^{2n+1}
\end{array}
$$

introduced in Section 2. A straightforward calculation of the mod-p homology Serre spectral sequence for the homotopy fibration $\Omega S^{2n+1} \longrightarrow F \longrightarrow P^{2n+1}(p^r)$ shows that $\widetilde{H}_*(F)$ is generated as a vector space by elements x_{2nk} in degrees $2nk$ for $k \geq 1$. Thus F has a natural filtration by skeleta. Let F_k be the $2nk$-skeleton of F. Then for each $k \geq 1$ there is a homotopy cofibration

$$
S^{2nk-1} \xrightarrow{\ g_{k-1}\ } F_{k-1} \longrightarrow F_k.
$$

The filtration on F lets us put a filtration on E. Define the spaces E_{k-1} and D_{k-1} by the homotopy pullback diagram

$$
(10) \qquad
\begin{array}{ccccc}
D_{k-1} & \longrightarrow & E_{k-1} & \longrightarrow & E \\
\| & & \downarrow & & \downarrow \\
D_{k-1} & \longrightarrow & F_{k-1} & \longrightarrow & F \\
& & \downarrow & & \downarrow \\
& & \Omega S^{2n+1} & = & \Omega S^{2n+1}.
\end{array}
$$

Since $F_{k-1} \longrightarrow F$ is a skeletal inclusion, a Serre spectral sequence calculation immediately implies that D_{k-1} is $(2nk-2)$-connected and has a single cell in dimension $2nk-1$. Further, the composite $S^{2nk-1} \hookrightarrow D_{k-1} \longrightarrow F_{k-1}$ is homotopic to g_{k-1}. Let $\overline{g}_{k-1}$ be the composite

$$
\overline{g}_{k-1} \colon S^{2nk-1} \hookrightarrow D_{k-1} \longrightarrow E_{k-1}.
$$

In Proposition 5.1 we will show that $\overline{g}_{k-1}$ is divisible by p^r, given mild conditions.

Proposition 5.1. *If $k > 1$ then there is a homotopy commutative diagram*

$$
\begin{array}{ccc}
S^{2nk-1} & \xrightarrow{\ \overline{g}_{k-1}\ } & E_{k-1} \\
\Big\downarrow{\underline{p}^r} & & \Big\| \\
S^{2nk-1} & \xrightarrow{\ \overline{h}_{k-1}\ } & E_{k-1}
\end{array}
$$

for some map $\overline{h}_{k-1}$.

Proposition 5.1 appeared in [GT] and relied heavily on the work of Cohen, Moore, and Neisendorfer [CMN1]. However, we will give a different proof which is self-contained. While some of the basic tools will be the same, we will require much less homological algebra and virtually none of the differential graded Lie algebra techniques from [CMN1] which underpinned the proof in [GT].

We begin with a small amount of mod-p homotopy theory; the primary reference is [N1]. In general, suppose we are given maps $f \colon P^s(p^r) \longrightarrow \Omega X$ and $g \colon P^t(p^r) \longrightarrow \Omega X$. As the smash $P^s(p^r) \wedge P^t(p^r)$ is homotopy equivalent to the wedge $P^{s+t}(p^r) \vee P^{s+t-1}(p^r)$, we can define the mod-$p^r$ Samelson product of f and g as the composite

$$\langle f, g \rangle \colon P^{s+t}(p^r) \longrightarrow P^s(p^r) \wedge P^t(p^r) \xrightarrow{\ [f,g]\ } \Omega X$$

where $[f, g]$ is the ordinary Samelson product of f and g. In our case, let $\nu \colon P^{2n}(p^r) \longrightarrow \Omega P^{2n+1}(p^r)$ be the adjoint of the identity map and let $\mu \colon S^{2n-1} \longrightarrow \Omega P^{2n+1}(p^r)$ be the inclusion of the bottom cell. Let $\mathfrak{ad}^0 = \mu$, $\mathfrak{ad}^1$ be the ordinary Samelson product $[\nu, \mu]$, and for $j > 1$ let $\mathfrak{ad}^{j-1} \colon P^{2nj-1}(p^r) \longrightarrow \Omega P^{2n+1}$ be the mod-p^r Samelson product defined inductively by $\mathfrak{ad}^{j-1} = \langle \nu, \mathfrak{ad}^{j-2} \rangle$.

Homologically, since $P^{2n+1}(p^r)$ is the suspension of $P^{2n}(p^r)$, the Bott-Samelson theorem implies that $H_*(\Omega P^{2n+1}(p^r)) \cong T(\widetilde{H}_*(P^{2n}(p^r)))$, where $T(\ )$ denotes the free tensor algebra. A basis for $\widetilde{H}_*(P^{2n}(p^r))$ is given by $\{u, v\}$ where u and v are in degrees $2n - 1$ and $2n$ respectively. As a matter of book-keeping, recall the well known fact that if V is a graded module over a field then $T(V) \cong UL\langle V \rangle$, where the right side is the universal enveloping algebra of the free Lie algebra generated by V. In our case, we have $H_*(\Omega P^{2n+1}(p^r)) \cong UL\langle u, v \rangle$. Picking out particular elements in $L\langle u, v \rangle$, let $ad^0 = u$ and for $j \geq 1$ define ad^j inductively by $ad^j = [v, ad^{j-1}]$.

In general, given a map $f \colon P^s(p^r) \longrightarrow \Omega X$, the mod-$p^r$ Hurewicz homomorphism is defined by sending f to $f_*(\iota)$, where ι is the generator in $H_s(P^s(p^r))$. In our case, the ordinary Hurewicz image of μ is u and the mod-p^r Hurewicz image of ν is v. The mod-p^r Hurewicz image of $P^{2nj-1}(p^r) \xrightarrow{\ \mathfrak{ad}^{j-1}\ } \Omega P^{2n+1}(p^r)$ is therefore ad^{j-1}.

With these ingredients in place, consider the homotopy fibration $\Omega E \longrightarrow \Omega P^{2n+1}(p^r) \xrightarrow{\ \Omega i\ } \Omega S^{2n+1}\{p^r\}$. For $j > 1$ the map $\mathfrak{ad}^{j-1}$ is defined via a Samelson product and so composes trivially

with the loop map Ωi since $\Omega S^{2n+1}\{p^r\}$ is homotopy commutative. Thus there is a lift

$$
\begin{array}{ccc}
 & & \Omega E \\
 & \nearrow^{l_{j-1}} & \downarrow \\
P^{2nk-1}(p^r) & \xrightarrow{\mathfrak{ad}^{j-1}} & \Omega P^{2n+1}(p^r)
\end{array}
$$

for some map l_{j-1}. Let l'_{j-1} be the composite $l'_{j-1}\colon P^{2nk-1}(p^r) \xrightarrow{l_{j-1}} \Omega E \longrightarrow \Omega F$. Then (9) implies that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
 & & \Omega F \\
 & \nearrow^{l'_{j-1}} & \downarrow \\
P^{2nk-1}(p^r) & \xrightarrow{\mathfrak{ad}^{j-1}} & \Omega P^{2n+1}(p^r).
\end{array}
$$

Note that for ΩF there is an additional lift. By connectivity, the map $S^{2n-1} \xrightarrow{\mathfrak{ad}^0} \Omega P^{2n+1}(p^r)$ lifts to a map $l'_0\colon S^{2n-1} \longrightarrow \Omega F$.

In particular, the Hurewicz image ad^0 of $\mathfrak{ad}^0$ factors through $H_*(\Omega F)$, as do the mod-p^r Hurewicz images ad^{j-1} of $\mathfrak{ad}^{j-1}$ for each $j > 1$. Since the skeletal inclusion $F_{k-1} \longrightarrow F$ is $(2nk-1)$-connected, for dimension and connectivity reasons the maps l'_{j-1} factor through ΩF_{k-1} for $0 \leq j-1 < k-1$. Thus the submodule $W_{k-1} = \{ad^{j-1}\}_{j=1}^{k-1}$ of Hurewicz images in $H_*(\Omega P^{2n+1}(p^r))$ factors through $H_*(\Omega F_{k-1})$. Extending multiplicatively gives a commative diagram

$$
\begin{array}{ccc}
T(W_{k-1}) & \longrightarrow & UL\langle u, v\rangle \\
\downarrow^{t_{k-1}} & & \downarrow^{\cong} \\
H_*(\Omega F_{k-1}) & \longrightarrow & H_*(\Omega P^{2n+1}(p^r))
\end{array}
$$

which defines the map t_{k-1}. Since $T(W_k) \cong UL\langle W_k\rangle$, and $L\langle W_k\rangle$ is a sub-Lie algebra of $L\langle u, v\rangle$, the Poincaré-Birkhoff-Witt Theorem implies that $UL\langle W_k\rangle$ is a subalgebra of $UL\langle u, v\rangle$. Thus the upper direction around the diagram is an injection, and so t_{k-1} is an injection. In the next Lemma we show that t_{k-1} is actually an isomorphism.

Lemma 5.2. *The map $T(W_{k-1}) \xrightarrow{t_{k-1}} H_*(\Omega F_{k-1})$ is an algebra isomorphism.*

Proof. Since t_{k-1} is an algebra injection, to prove the lemma it suffices to show that $T(W_{k-1})$ and $H_*(\Omega F_{k-1})$ have the same Euler-Poincaré series. Denote the Euler-Poincaré series of a graded module A by $P(A)$. As t_{k-1} is an injection, we have $P(T(W_{k-1})) \leq P(H_*(\Omega F_{k-1}))$. On the other hand, the cobar construction implies that there is an isomorphism $H_*(\Omega F_{k-1}) \cong H_*(T(V_{k-1}), d)$, where $V_{k-1} = \Sigma^{-1} H_*(F_{k-1})$ is a desuspension of the module $H_*(F_{k-1})$, d is an appropriate differential on the tensor algebra, and homology has been taken with respect to that differential. In particular, the Euler-Poincaré series $P(H_*(T(V_{k-1}), d))$ is bounded above by $P(T(V_{k-1}))$. Next, since F_{k-1} has a single cell in dimension $2nj$ for each $1 \leq j \leq k-1$, a basis for V_{k-1} is $\{x_{2nj-1}\}_{j=1}^{k-1}$, where x_{2nj-1} is

in degree $2nj-1$. Observe that there is an abstract isomorphism of graded modules $W_{k-1} = V_{k-1}$ and so $P(T(V_{k-1})) = P(T(W_{k-1}))$. Putting all this together gives inequalities

$$P(T(W_{k-1})) \leq P(H_*(\Omega F_{k-1})) = P(H_*((T(V_{k-1}), d)) \leq P(T(V_{k-1})) = P(T(W_{k-1}))$$

and so $P(T(W_{k-1}) = P(H_*(\Omega F_{k-1}))$, as required. $\qquad\square$

Inductively, we obtain the following

Corollary 5.3. *There is an isomorphism $H_*(\Omega F) \cong T(W_\infty)$, where $W_\infty = \{ad^{j-1}\}_{j=1}^\infty$, and the map $\Omega F_{k-1} \longrightarrow \Omega F$ is modelled homologically by the inclusion of tensor algebras $T(W_{k-1}) \longrightarrow T(W_\infty)$.* $\qquad\square$

Corollary 5.3 implies that the mod-p^r Hurewicz image of $P^{2nk-1}(p^r) \xrightarrow{l'_{k-1}} \Omega F$ can be regarded as ad^{k-1}. By (10), there is a homotopy fibration $\Omega F_{k-1} \longrightarrow \Omega F \longrightarrow D_{k-1}$. Recall that D_{k-1} is $(2nk-2)$-connected and has a single cell in dimension $2nk-1$. The following lemma implies that in homology ad^{k-1} is mapped to the generator in $H_{2nk-1}(D_{k-1})$.

Lemma 5.4. *The composite $P^{2nk-1}(p^r) \xrightarrow{l'_{k-1}} \Omega F \longrightarrow D_{k-1}$ is degree one in $H_{2nk-1}(\ )$.*

Proof. By Corollary 5.3, the map $\Omega F_{k-1} \longrightarrow \Omega F$ is modelled homologically by the inclusion of tensor algebras $T(W_{k-1}) \longrightarrow T(W_\infty)$. Since $W_\infty = W_{k-1} \oplus V$, where $V = \{ad^{j-1}\}_{j=k}^\infty$, there is a free product decomposition $T(W_\infty) \cong T(W_{k-1}) \coprod T(V)$. In particular, this implies that there is a decomposition $T(W_\infty) \cong T(W_{k-1}) \otimes N$ of left $T(W_{k-1})$-modules for some module N. Consequently, the Eilenberg-Moore spectral sequence for the homotopy fibration $\Omega F_{k-1} \longrightarrow \Omega F_k \longrightarrow D_{k-1}$ which converges to $H_*(D_{k-1})$ collapses at E^2. That is,

$$E^2 = \operatorname{Tor}^{T(W_k)}(\mathbb{Z}/p\mathbb{Z}, T(W_\infty)) = \mathbb{Z}/p\mathbb{Z} \otimes_{T(W_k)} T(W_\infty) = N,$$

and so $H_*(D_{k-1}) \cong N$. This implies that the element $ad^{k-1} \in V \cap N$ is in $H_*(D_{k-1})$. But ad^{k-1} is the mod-p^r Hurewicz image of the map $P^{2nk-1}(p^r) \xrightarrow{l'_{k-1}} \Omega F$, and so the composite $P^{2nk-1}(p^r) \xrightarrow{l'_{k-1}} \Omega F \longrightarrow D_{k-1}$ is degree one in $H_{2nk-1}(\ )$. $\qquad\square$

We are now ready to prove Proposition 5.1

Proof of Proposition 5.1: We begin with some geometric arguments to reduce the proof to a homology calculation. Define the space M_{k-1} by the homotopy pullback diagram

$$
(11) \qquad
\begin{array}{ccccc}
D_{k-1} & \longrightarrow & M_{k-1} & \longrightarrow & \Omega S^{2n+1}\{p^r\} \\
\| & & \downarrow & & \downarrow \\
D_{k-1} & \longrightarrow & E_{k-1} & \longrightarrow & E \\
& & \downarrow & & \downarrow \\
& & P^{2n+1}(p^r) & = & P^{2n+1}(p^r).
\end{array}
$$

Taking vertical connecting maps and reorienting, we obtain a homotopy pullback diagram

(12)

$$
\begin{array}{ccccc}
\Omega E & \xrightarrow{\ \partial_1\ } & D_{k-1} & \longrightarrow & E_{k-1} \\
\downarrow & & \downarrow & & \| \\
\Omega P^{2n+1}(p^r) & \longrightarrow & M_{k-1} & \longrightarrow & E_{k-1} \\
{\scriptstyle \Omega i}\downarrow & & \downarrow & & \\
\Omega S^{2n+1}\{p^r\} & = = & \Omega S^{2n+1}\{p^r\} & &
\end{array}
$$

which defines the map ∂_1.

By definition, the map $P^{2nk-1}(p^r) \xrightarrow{l_{k-1}} \Omega E$ is a lift of $P^{2nk-1}(p^r) \xrightarrow{\mathfrak{a}\mathfrak{d}^{j-1}} \Omega P^{2n+1}(p^r)$. Let λ' be the composite

$$
\lambda' \colon P^{2nk-1}(p^r) \xrightarrow{l_{k-1}} \Omega E \xrightarrow{\partial_1} D_{k-1}.
$$

Since D_{k-1} is $(2nk-2)$-connected, the restriction of λ' to the $(2nk-2)$-cell of $P^{2nk-1}(p^r)$ is null homotopic, implying that λ' factors as a composite $P^{2nk-1}(p^r) \xrightarrow{q} S^{2nk-1} \xrightarrow{\lambda} D_{k-1}$, for some map λ.

Consider the diagram

(13)

$$
\begin{array}{ccccc}
P^{2nk-1}(p^r) & \xrightarrow{\ q\ } & S^{2nk-1} & \xrightarrow{\ \underline{p}^r\ } & S^{2nk-1} \\
{\scriptstyle l_{k-1}}\downarrow & & {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle h_{k-1}} \\
\Omega E & \xrightarrow{\ \partial_1\ } & D_{k-1} & \longrightarrow & E_{k-1}.
\end{array}
$$

The conclusion of the previous paragraph implies that the left square homotopy commutes. Since the lower row is a homotopy fibration, the homotopy commutativity of the left square implies that $\lambda \circ q$ composes trivially into E_{k-1}. Thus there is an extension along the homotopy cofiber of q which makes the right square homotopy commute for some map h_{k-1}.

We claim that the composite $\partial_1 \circ l_{k-1}$ is degree one in $H_{2nk-1}(\ )$. Suppose this is the case. Then the left square in (13) implies that λ is degree one in $H_{2nk-1}(\ )$. Thus λ is homotopic to the inclusion of the bottom cell into D_{k-1}. So by the definition of $\overline{g}_{k-1}$, the composite $S^{2nk-1} \xrightarrow{\lambda} D_{k-1} \longrightarrow M_{k-1} \longrightarrow E_{k-1}$ is homotopic to $\overline{g}_{k-1}$. Thus the right square in (13) implies that $\overline{g}_{k-1} \simeq h_{k-1} \circ \underline{p}^r$, as asserted by the proposition.

It remains to show that $\partial_1 \circ l_{k-1}$ is degree one in $H_{2nk-1}(\ )$. Refining a bit, by (10) there is an extended homotopy pullback diagram

$$
\begin{array}{ccccccc}
\Omega E & \xrightarrow{\ \partial_1\ } & D_{k-1} & \longrightarrow & E_{k-1} & \longrightarrow & E \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\Omega F & \xrightarrow{\ \partial_2\ } & D_{k-1} & \longrightarrow & F_{k-1} & \longrightarrow & F
\end{array}
$$

which defines the map ∂_2. By definition, l'_{k-1} is the composite $P^{2nk-1}(p^r) \xrightarrow{l_{j-1}} \Omega E \longrightarrow \Omega F$. So to show that $\partial_1 \circ l_{k-1}$ is degree one in $H_{2nk-1}(\ )$ it is equivalent to show that $\partial_2 \circ l'_{k-1}$ is degree one in $H_{2nk-1}(\ )$. But this is the case by Lemma 5.4. $\square$

6. p^r-DIVISIBILITY II

The second p^r-divisibility property concerns the map $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}$. Since $H_*(\Omega S^{2n+1}) \cong \mathbb{Z}/p\mathbb{Z}[x_{2n}]$, there is a natural filtration on ΩS^{2n+1} by skeleta. For $k \geq 0$, let $J_k(S^{2n})$ be the $2nk$-skeleton of ΩS^{2n+1}. Then for each $k \geq 1$ there is a homotopy cofibration

$$S^{2nk-1} \xrightarrow{s_{k-1}} J_{k-1}(S^{2n}) \longrightarrow J_k(S^{2n}).$$

The filtation on ΩS^{2n+1} lets us put a filtration on $\Omega P^{2n+1}(p^r)$. For $k \geq 0$ define the spaces P_{k-1} and B_{k-1}, and the map γ_{k-1}, by the iterated homotopy pullback

$$
\begin{array}{ccccc}
B_{k-1} & \longrightarrow & P_{k-1} & \xrightarrow{\gamma_{k-1}} & \Omega P^{2n+1}(p^r) \\
\| & & \downarrow & & \downarrow{\scriptstyle \Omega q} \\
B_{k-1} & \longrightarrow & J_{k-1}(S^{2n}) & \longrightarrow & \Omega S^{2n+1}.
\end{array}
$$

Since $J_{k-1}(S^{2n}) \longrightarrow \Omega S^{2n+1}$ is a skeletal inclusion, a Serre spectral sequence calculation immediately implies that B_{k-1} is $(2nk-2)$-connected and has a single cell in dimension $2nk-1$. Further, the composite $S^{2nk-1} \hookrightarrow B_{k-1} \longrightarrow J_{k-1}(S^{2n})$ is homotopic to s_{k-1}. Let p_{k-1} be the composite

$$p_{k-1} \colon S^{2nk-1} \hookrightarrow B_{k-1} \longrightarrow P_{k-1}.$$

In Proposition 6.1 we show that the map p_{k-1} is divisible by p^r after suspending.

Proposition 6.1. *For $k \geq 1$, there is a homotopy commutative diagram*

$$
\begin{array}{ccc}
\Sigma S^{2nk-1} & \xrightarrow{\Sigma p_{k-1}} & \Sigma P_{k-1} \\
\downarrow{\scriptstyle \Sigma \underline{p^r}} & & \| \\
\Sigma S^{2nk-1} & \xrightarrow{\overline{p}_{k-1}} & \Sigma P_{k-1}
\end{array}
$$

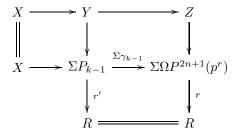*for some map $\overline{p}_{k-1}$.*

Proof. Since $P^{2n+1}(p^r) \simeq \Sigma P^{2n}(p^r)$, the Bott-Samelson Theorem implies that $H_*(\Omega P^{2n+1}(p^r)) \cong T(\widetilde{H}_*(P^{2n}(p^r))) \cong T(u,v)$, where u and v are in degrees $2n-1$ and $2n$ respectively, and the action of the Bockstein is determined by $\beta^r(v) = u$. In particular, $\beta^r(v^k) \neq 0$. We will focus on the pair $(v^k, \beta^r(v^k))$.

Since $P^{2n+1}(p^r) \simeq \Sigma P^{2n}(p^r)$, the James construction [J] implies that there is a homotopy equivalence $\Sigma \Omega P^{2n+1}(p^r) \simeq \bigvee_{i=1}^{\infty} \Sigma(P^{2n}(p^r))^{(i)}$, where $(P^{2n})^{(i)}$ is the i-fold smash of $P^{2n}(p^r)$ with itself. As the smash of two mod-p^r Moore spaces is homotopy equivalent to a wedge of two mod-p^r Moore spaces, iterating shows that $(P^{2n}(p^r))^{(i)}$ is homotopy equivalent to a wedge of mod-p^r Moore spaces

for each $i \geq 2$, and hence so is $\Sigma\Omega P^{2n+1}(p^r)$. In particular, there is an inclusion of a wedge summand $j\colon P^{2nk+1}(p^r) \longrightarrow \Sigma\Omega P^{2n+1}(p^r)$ whose image in homology is the pair $\{\sigma v^k, \sigma\beta^r(v^k)\}$. Define the space R and the map r by the homotopy cofibration

$$P^{2np+1}(p^r) \xrightarrow{j} \Sigma\Omega P^{2n+1}(p^r) \xrightarrow{r} R.$$

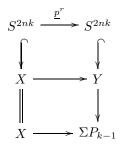Define spaces X, Y, and Z by the homotopy pullback diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\Big\| & & \Big\downarrow & & \Big\downarrow \\
X & \longrightarrow & \Sigma P_{k-1} & \xrightarrow{\Sigma\gamma_{k-1}} & \Sigma\Omega P^{2n+1}(p^r) \\
& & \Big\downarrow{\scriptstyle r'} & & \Big\downarrow{\scriptstyle r} \\
& & R & =\!=\!= & R
\end{array}
$$

where r' is defined as the composite $r \circ \Sigma\gamma_{k-1}$. We aim to identify X, Y, and Z through a skeletal range. This will involve repeated applications of the Blakers-Massey Theorem, or equivalently, the Serre exact sequence. These state that if A is $(m-1)$-connected and C is $(n-1)$-connected then in dimensions $\leq n + m - 1$ the sequence $A \longrightarrow B \longrightarrow C$ is a homotopy fibration if and only if it is a homotopy cofibration.

First, consider the homotopy fibration $Z \longrightarrow \Sigma\Omega P^{2n+1}(p^r) \xrightarrow{r} R$. By its definition, R is $(2n-1)$-connected, so by the Blakers-Massey Theorem, the homotopy cofibration $P^{2nk+1}(p^r) \xrightarrow{j} \Sigma\Omega P^{2n+1}(p^r) \xrightarrow{r} R$ is a homotopy fibration in dimensions $\leq 2nk + 2n - 1$. But Z is defined to be the homotopy fiber of r, so $Z \simeq P^{2nk+1}(p^r)$ in dimensions $\leq 2nk + 2n - 1$.

Second, before considering $\Sigma\gamma_{k-1}$ we take another look at γ_{k-1} and the homotopy fibration $B_{k-1} \longrightarrow P_{k-1} \xrightarrow{\gamma_{k-1}} \Omega P^{2n+1}(p^r)$. Recall that the $(2nk-1)$-skeleton of B_{k-1} is S^{2nk-1} and p_{k-1} is the composite $S^{2nk-1} \hookrightarrow B_{k-1} \longrightarrow P_{k-1}$. Since $\Omega P^{2n+1}(p^r)$ is $(2n-2)$-connected, the Blakers-Massey Theorem implies that the sequence $S^{2nk-1} \xrightarrow{p_{k-1}} P_{k-1} \xrightarrow{\gamma_{k-1}} \Omega P^{2n+1}(p^r)$ is a homotopy cofibration in dimensions $\leq 2nk + 2n - 3$. Suspending, the sequence $S^{2nk} \xrightarrow{\Sigma p_{k-1}} \Sigma P_{k-1} \xrightarrow{\Sigma\gamma_{k-1}} \Sigma\Omega P^{2n+1}(p^r)$ is a homotopy cofibration in dimensions $\leq 2nk + 2n - 1$. By the Blakers-Massey Theorem this sequence is also a homotopy fibration in dimensions $\leq 2nk + 2n - 1$. On the other hand, the homotopy fiber of $\Sigma\gamma_{k-1}$ is defined to be X, so $X \simeq S^{2nk}$ in dimensions $\leq 2nk + 2n - 1$, and the composite $S^{2nk} \hookrightarrow X \longrightarrow \Sigma P_{k-1}$ is homotopic to Σp_{k-1}.

Third, consider the homotopy fibration $X \longrightarrow Y \longrightarrow Z$. In dimensions $\leq 2nk + 2n - 1$ we have $X \simeq S^{2nk}$ and $Z \simeq P^{2nk+1}(p^r)$. The Blakers-Massey Theorem implies that this homotopy fibration is also a homotopy cofibration in this dimensional range. That is, in dimensions $\leq 2nk + 2n - 1$, there is a homotopy cofibration $S^{2nk} \xrightarrow{a} Y \longrightarrow P^{2nk+1}(p^r)$ for some map a. The corresponding long exact sequence in homology implies that $H_*(Y) \cong H_*(S^{2nk})$ and $a_* = (p^r)_*$ in dimension $\leq 2nk + 2n - 1$. Thus $Y \simeq S^{2nk}$ and $a \simeq \underline{p}^r$ in dimensions $\leq 2nk + 2n - 1$.

Combining the conclusions of the previous two paragraphs gives a homotopy commutative diagram

$$
\begin{array}{ccc}
S^{2nk} & \xrightarrow{\;\underline{p^r}\;} & S^{2nk} \\
\cup \downarrow & & \downarrow \cup \\
X & \longrightarrow & Y \\
\Big\| & & \downarrow \\
X & \longrightarrow & \Sigma P_{k-1}
\end{array}
$$

in which the lower direction around the diagram is homotopic to p_{k-1}. The outer rectangle is therefore the diagram asserted by the Proposition. □

Remark 6.2. When $k = 1$ the divisibility property in Proposition 6.1 actually holds before suspending. This is seen directly from the homotopy fibration $B_0 \longrightarrow P_0 \longrightarrow \Omega P^{2n+1}(p^r)$ which, by the Blakers-Massey Theorem, is the homotopy cofibration $S^{2n-1} \xrightarrow{\;p^r\;} S^{2n-1} \longrightarrow P^{2n}(p^r)$ in dimensions $\leq 4n - 3$.

7. EXTENSIONS

In this section we prove Propositions 2.1 and 2.2, which will complete the proof of Theorem 1.1.

Proof of Proposition 2.1: Filter E and F by the spaces E_k and F_k considered in Section 5, so $E = \varinjlim E_k$ and $F = \varinjlim F_k$. By (10), there are homotopy fibration sequences $\Omega^2 S^{2n+1} \xrightarrow{\partial_k} E_k \xrightarrow{f_k} F_k \longrightarrow \Omega S^{2n+1}$. Note that when $k = 0$ we have $F_0 \simeq *$, so $E_0 \simeq \Omega^2 S^{2n+1}$ and ∂_0 is the identity map. Let $\epsilon_0 \colon E_0 \longrightarrow \Omega^2 S^{2np+1}\{p\}$ be the map $\Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{2np+1}\{p\}$. The asserted extension $E \xrightarrow{e_1} \Omega^2 S^{2np+1}\{p\}$ of S will be constructed as the limit of a sequence of extensions $E_k \xrightarrow{\epsilon_k} \Omega^2 S^{2np+1}\{p\}$, where ϵ_k extends ϵ_{k-1}. There are two cases: $k = 1$ and $k > 1$.

The setup for both cases is the same. From the homotopy cofibration $S^{2nk-1} \xrightarrow{g_{k-1}} F_{k-1} \longrightarrow F_k$ and the map $E_k \xrightarrow{f_k} F_k$ we obtain a diagram of iterated homotopy pullbacks

$$
\begin{array}{ccccc}
\Omega^2 S^{2n+1} & \!=\!\!=\! & \Omega^2 S^{2n+1} & \!=\!\!=\! & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow{\scriptstyle \partial_k} & & \downarrow{\scriptstyle \partial_{k-1}} \\
QE_{k-1} & \longrightarrow & E_{k-1} & \longrightarrow & E_k \\
\downarrow & & \downarrow{\scriptstyle f_{k-1}} & & \downarrow{\scriptstyle f_k} \\
S^{2nk-1} & \xrightarrow{\;g_{k-1}\;} & F_{k-1} & \longrightarrow & F_k
\end{array}
$$

which defines the space QE_{k-1}. Lemma 4.1 implies that $QE_{k-1} \simeq S^{2nk-1} \times \Omega^2 S^{2n+1}$ and there is a homotopy pushout

(14)

$$
\begin{array}{ccc}
S^{2nk-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\theta_{k-1}} & E_{k-1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{\partial_k} & E_k
\end{array}
$$

for some map θ_{k-1}. As the homotopy fibration $\Omega^2 S^{2n+1} \xrightarrow{\partial_{k-1}} E_{k-1} \xrightarrow{f_{k-1}} F_{k-1}$ is principal, by Remark 4.2, θ_{k-1} can be taken to be the composite $S^{2nk-1} \times \Omega^2 S^{2n+1} \xrightarrow{a \times 1} E_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{\vartheta_{k-1}} E_{k-1}$, where a is any choice of a lift of g_{k-1} to E_{k-1} and ϑ_{k-1} is the homotopy action from the principal fibration.

Suppose $k = 1$. Then $E_0 \simeq \Omega^2 S^{2n+1}$ and the action ϑ_0 is the loop space multiplication. We claim that a must be E^2, implying that $\theta_0 \simeq E^2 \cdot 1$. To see this, observe that in the homotopy fibration $E_1 \xrightarrow{f_1} F_1 \longrightarrow \Omega S^{2n+1}$ we have $F_1 \simeq S^{2n}$ and the right map is degree one. Thus E_1 is $(4n-3)$-connected. Therefore the restriction of θ_0 in (14) must be degree one, that is, $a \simeq E^2$. Thus we have a homotopy pushout

$$
\begin{array}{ccc}
S^{2n-1} \times \Omega^2 S^{2n+1} & \xrightarrow{E^2 \cdot 1} & \Omega^2 S^{2n+1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \partial_1} \\
\Omega^2 S^{2n+1} & \xrightarrow{\partial_1} & E_1.
\end{array}
$$

By Lemma 3.3, there is a homotopy commutative diagram

$$
\begin{array}{ccc}
S^{2n-1} \times \Omega^2 S^{2n+1} & \xrightarrow{E^2 \cdot 1} & \Omega^2 S^{2n+1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle S} \\
\Omega^2 S^{2n+1} & \xrightarrow{S} & \Omega^2 S^{2np+1}\{p\}.
\end{array}
$$

Hence there is a pushout map $\epsilon_1 : E_1 \longrightarrow \Omega^2 S^{2np+1}\{p\}$ with the property that ϵ_1 extends $\epsilon_0 = S$.

Next, suppose $k > 1$ and assume that there is a map $\epsilon_{k-1} : E_{k-1} \longrightarrow \Omega^2 S^{2np+1}\{p\}$ with the property that ϵ_{k-1} extends ϵ_{k-2}. In (14) we have seen that θ_{k-1} factors as $S^{2nk-1} \times \Omega^2 S^{2n+1} \xrightarrow{a \times 1} E_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{\vartheta_{k-1}} E_{k-1}$, where a lifts g_{k-1}. By Lemma 5.1, we can choose a to be $\overline{g}_{k-1}$, which has the property that $\overline{g}_{k-1} \simeq \overline{h}_{k-1} \circ \underline{p}^r$ for some map $\overline{h}_{k-1}$. This fulfils one hypothesis of Theorem 4.3. The other is fulfilled because the H-space $\Omega^2 S^{2np+1}\{p\}$ is homotopy associative and by [N3] its pth-power map is null homotopic. Thus Theorem 4.3 implies that there is an extension

$$
\begin{array}{ccc}
E_{k-1} & \xrightarrow{\epsilon_{k-1}} & \Omega^2 S^{2np+1}\{p\} \\
\downarrow & & \| \\
E_k & \xrightarrow{\epsilon_k} & \Omega^2 S^{2np+1}\{p\}
\end{array}
$$

for some map ϵ_k. $\square$

As described in (5), the extension in Proposition 2.1 implies there is a homotopy fibration sequence $\Omega^2 S^{2n+1} \longrightarrow Y \longrightarrow X \xrightarrow{q_X} \Omega S^{2n+1}$, and in Proposition 2.2 we want to extend the map $Y \xrightarrow{g} W_{np}$ to a map $X \longrightarrow W_{np}$. To do this we will filter X and produce iterated extensions using Theorem 4.3, but first a certain decomposition issue needs to be resolved.

For $k \geq 0$, define the space X_k and the map q_k by the homotopy pullback diagram

$$
\begin{array}{ccc}
X_k & \longrightarrow & X \\
\downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle q_X} \\
J_k(S^{2n}) & \longrightarrow & \Omega S^{2n+1}.
\end{array}
$$

From the homotopy cofibration $S^{2nk-1} \xrightarrow{s_{k-1}} J_{k-1}(S^{2n}) \longrightarrow J_k(S^{2n})$ and the map $X_k \longrightarrow J_k(S^{2n})$, we obtain a diagram of iterated homotopy pullbacks

(15)
$$
\begin{array}{ccccc}
Y & = & Y & = & Y \\
\downarrow & & \downarrow & & \downarrow \\
QX_{k-1} & \longrightarrow & X_{k-1} & \longrightarrow & X_k \\
\downarrow & & \downarrow{\scriptstyle q_{k-1}} & & \downarrow{\scriptstyle q_k} \\
S^{2nk-1} & \xrightarrow{s_{k-1}} & J_{k-1}(S^{2n}) & \longrightarrow & J_k(S^{2n})
\end{array}
$$

which defines the space QX_{k-1}. Lemma 4.1 implies that $QX_{k-1} \simeq S^{2nk-1} \times Y$ and there is a homotopy pushout

$$
\begin{array}{ccc}
QX_{k-1} \simeq S^{2nk-1} \times Y & \xrightarrow{\theta_{k-1}} & X_{k-1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
Y & \xrightarrow{\delta_k} & X_k
\end{array}
$$

for some map θ_{k-1}. However, the fibration $Y \longrightarrow X_{k-1} \xrightarrow{q_{k-1}} J_{k-1}(S^{2n})$ may not be principal so we cannot appeal to Remark 4.2 in order to show that the equivalence for QX_{k-1} can be chosen so that θ_{k-1} factors in the manner required by Theorem 4.3. To obtain such a factorization we have to work a bit harder, and this is the purpose of the next three lemmas.

Recall from (6) that the map $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}$ factors as the composite $\Omega P^{2n+1}(p^r) \xrightarrow{i_X} X \longrightarrow \Omega S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}$. So for each $k \geq 1$ there is an iterated homotopy pullback diagram

(16)
$$
\begin{array}{ccccccc}
P_{k-1} & \longrightarrow & X_{k-1} & \longrightarrow & C_{k-1} & \longrightarrow & J_{k-1}(S^{2n}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega P^{2n+1}(p^r) & \xrightarrow{i_X} & X & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \longrightarrow & \Omega S^{2n+1}
\end{array}
$$

which defines the space C_{k-1}. Observe that the left and middle homotopy pullbacks in (16) give a rectangle which is a homotopy pullback. In particular, this implies that there is a homotopy fibration $P_{k-1} \longrightarrow C_{k-1} \longrightarrow E$. Also, the right homotopy pullback implies that there is a homotopy fibration $C_{k-1} \longrightarrow J_{k-1}(S^{2n}) \xrightarrow{j_{k-1}} \Omega S^{2n+1}$, where j_{k-1} is the composite $j_{k-1}\colon J_{k-1}(S^{2n}) \hookrightarrow \Omega S^{2n+1} \xrightarrow{p^r}$

ΩS^{2n+1}. In particular, there is a homotopy action $C_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1}$. Let u be the composite

$$u : P_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1}.$$

Lemma 7.1. *There is a map* $P_{k-1} \times Y \longrightarrow X_{k-1}$ *and a homotopy pullback*

$$
\begin{array}{ccc}
P_{k-1} \times Y & \longrightarrow & X_{k-1} \\
\downarrow & & \downarrow \\
P_{k-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\ u\ } & C_{k-1}.
\end{array}
$$

Proof. Consider the diagram of extended homotopy pullbacks

$$
\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \longrightarrow & C_{k-1} & \longrightarrow & J_{k-1}(S^{2n}) & \xrightarrow{j_{k-1}} & \Omega S^{2n+1} \\
\| & & \downarrow & & \downarrow & & \| \\
\Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{p^r} & \Omega S^{2n+1} \\
\| & & \downarrow & & \downarrow & & \| \\
\Omega^2 S^{2n+1} & \longrightarrow & E & \longrightarrow & F & \longrightarrow & \Omega S^{2n+1}.
\end{array}
$$

Here, the lower ladder is from the definitions of the spaces E and F in (4) and the upper ladder is from the definition of the space C_{k-1}. The naturality of the canonical homotopy action for a fibration sequence implies that there is a homotopy commutative diagram of actions

$$
(17) \qquad
\begin{array}{ccc}
C_{k-1} \times \Omega^2 S^{2n+1} & \longrightarrow & C_{k-1} \\
\downarrow & & \downarrow \\
\Omega S^{2n+1}\{p^r\} \times \Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow \\
E \times \Omega^2 S^{2n+1} & \longrightarrow & E.
\end{array}
$$

Combining this with the homotopy fibration $P_{k-1} \longrightarrow C_{k-1} \longrightarrow E$ gives a homotopy commutative diagram

$$
(18) \qquad
\begin{array}{ccccc}
P_{k-1} \times \Omega^2 S^{2n+1} & \longrightarrow & C_{k-1} \times \Omega^2 S^{2n+1} & \longrightarrow & C_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
* \times \Omega^2 S^{2n+1} & \longrightarrow & E \times \Omega^2 S^{2n+1} & \longrightarrow & E.
\end{array}
$$

Note that the upper row of this diagram is the definition of u. Now compose the diagram with the map $E \xrightarrow{e_1} \Omega^2 S^{2np+1}\{p\}$. Let v and w be the composites

$$v : C_{k-1} \longrightarrow E \xrightarrow{e_1} \Omega^2 S^{2np+1}\{p\}$$

$$w : P_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{\ u\ } C_{k-1} \xrightarrow{\ v\ } \Omega^2 S^{2np+1}\{p\}.$$

We wish to identify the homotopy fibers of v and w. First, v factors as the composite $C_{k-1} \longrightarrow \Omega S^{2n+1}\{p^r\} \longrightarrow E \xrightarrow{e_1} \Omega^2 S^{2np+1}\{p\}$. The latter two maps in this composite define ϵ_1, whose homotopy fiber is X. Thus the middle pullback in (16) implies that the homotopy fiber of v is X_{k-1}. Second, the homotopy commutativity of (18) implies that w is homotopic to the composite $P_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{\pi_2} \Omega^2 S^{2n+1} \longrightarrow E \xrightarrow{e_1} \Omega^2 S^{2np+1}\{p\}$, which in turn is homotopic to the composite $P_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{\pi_2} \Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{2np+1}\{p\}$. As the homotopy fiber of S is Y, this implies that there is a homotopy fibration $P_{k-1} \times Y \longrightarrow P_{k-1} \times \Omega^2 S^{2n+1} \xrightarrow{w} \Omega^2 S^{2np+1}\{p\}$. Putting the fibrations for v and w together, we obtain a homotopy pullback diagram

$$
\begin{array}{ccc}
P_{k-1} \times Y & \longrightarrow & X_{k-1} \\
\downarrow & & \downarrow \\
P_{k-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\ u\ } & C_{k-1} \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} \\
\Omega^2 S^{2np+1}\{p\} & = & \Omega^2 S^{2np+1}\{p\}
\end{array}
$$

which proves the lemma. $\qquad\qquad\square$

Define $\overline{\theta}_{k-1}$ as the composite

$$
\overline{\theta}_{k-1} \colon S^{2nk-1} \times Y \xrightarrow{p_{k-1} \times 1} P_{k-1} \times Y \longrightarrow X_{k-1}.
$$

Lemma 7.2. *There is a homotopy pullback*

$$
\begin{array}{ccc}
S^{2nk-1} \times Y & \xrightarrow{\ \overline{\theta}_{k-1}\ } & X_{k-1} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle q_{k-1}} \\
S^{2nk-1} & \xrightarrow{\ s_{k-1}\ } & J_{k-1}(S^{2n}).
\end{array}
$$

Proof. Consider the diagram

$$
(19) \qquad
\begin{array}{ccccc}
S^{2nk-1} \times Y & \xrightarrow{p_{k-1} \times 1} & P_{k-1} \times Y & \longrightarrow & X_{k-1} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow & & \downarrow \\
 & & P_{k-1} \times \Omega^2 S^{2n+1} & \xrightarrow{\ u\ } & C_{k-1} \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\
S^{2nk-1} & \xrightarrow{\ p_{k-1}\ } & P_{k-1} & \longrightarrow & J_{k-1}(S^{2n}).
\end{array}
$$

The left rectangle is a homotopy pullback by the naturality of the projection. The upper right square is a homotopy pullback by Lemma 7.1. For the lower right square, by definition, u is the composite $P_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1}$ where the right map is the action associated to

the homotopy fibration $C_{k-1} \longrightarrow J_{k-1}(S^{2n}) \xrightarrow{j_{k-1}} \Omega S^{2n+1}$. A canonical property of such an action is that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
C_{k-1} \times \Omega^2 S^{2n+1} & \longrightarrow & C_{k-1} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow \\
C_{k-1} & \longrightarrow & J_{k-1}(S^{2n}).
\end{array}
$$

Note that this is in fact a homotopy pullback. When precomposed with the map $P_{k-1} \times \Omega^2 S^{2n+1} \longrightarrow C_{k-1} \times \Omega^2 S^{2n+1}$, the naturality of the projection π_1 implies that we obtain the lower right square in (19), and that this square is a homotopy pullback. Thus (19) consists of homotopy pullbacks and so its outer perimeter is also a homotopy pullback. As the top row of (19) is the definition of $\overline{\theta}_{k-1}$ and its right column is q_{k-1}, the outer perimeter of (19) is the diagram asserted by the lemma. $\square$

Lemma 7.2 describes the homotopy pullback of $X_{k-1} \longrightarrow J_{k-1}(S^{2n})$ and $S^{2nk-1} \xrightarrow{s_{k-1}} J_{k-1}(S^{2n})$. But by (15), QX_{k-1} is defined to be this pullback. Thus there is a homotopy equivalence $S^{2nk-1} \times Y \xrightarrow{\simeq} QX_{k-1}$ with an appropriate action property, as stated in the following.

Lemma 7.3. *There is a decomposition $QX_{k-1} \simeq S^{2nk-1} \times Y$ and a homotopy pullback*

$$
\begin{array}{ccccc}
QX_{k-1} \simeq S^{2nk-1} \times Y & \xrightarrow{p_{k-1} \times 1} & P_{k-1} \times Y & \longrightarrow & X_{k-1} \\
\downarrow{\scriptstyle \pi_1} & & & & \downarrow \\
S^{2nk-1} & \xrightarrow{\quad\quad s_{k-1} \quad\quad} & & & J_{k-1}(S^{2n})
\end{array}
$$

where p_{k-1} is the map in Proposition 6.1. In particular, for $k \geq 1$ we have $\Sigma p_{k-1} \simeq \overline{p}_{k-1} \circ \underline{p}^r$ for some map $\overline{p}_{k-1}$. $\square$

We are now ready to prove the existence of the second extension.

Proof of Proposition 2.2: Filter X and ΩS^{2n+1} by the spaces X_k and $J_k(S^{2n})$, so $X = \varinjlim X_k$ and $\Omega S^{2n+1} = \varinjlim J_k(S^{2n})$. By (15), there are homotopy fibrations $Y \xrightarrow{\delta_k} X_k \xrightarrow{q_k} J_k(S^{2n})$. Note that when $k = 0$ we have $J_0(S^{2n}) \simeq *$, so $X_0 \simeq Y$ and δ_0 is the identity map. Let $\varepsilon_0 \colon X_0 \longrightarrow W_{np}$ be the map $Y \xrightarrow{g} W_{np}$. The asserted extension $X \xrightarrow{e_2} W_{np}$ of g will be constructed as the limit of a sequence of extensions $X_k \xrightarrow{\varepsilon_k} \Omega W_{np}$ with the property that ε_k extends ε_{k-1}.

If $k = 1$ we have $X_0 \xrightarrow{\varepsilon_0} \Omega W_{np}$ in place. If $k > 1$ assume that there is a map $\varepsilon_{k-1} \colon X_{k-1} \longrightarrow \Omega W_{np}$ with the property that ε_{k-1} is an extension of ε_{k-2}. We set up to use Theorem 4.3. From the homotopy cofibration $S^{2nk-1} \xrightarrow{s_{k-1}} J_{k-1}(S^{2n}) \longrightarrow J_k(S^{2n})$ and the map $X_k \xrightarrow{q_k} J_k(S^{2n})$ we obtain

a diagram of iterated homotopy pullbacks

$$
\begin{array}{ccccc}
Y & =\!=\!=\!= & Y & =\!=\!=\!= & Y \\
\downarrow & & \downarrow{\scriptstyle \delta_{k-1}} & & \downarrow{\scriptstyle \delta_k} \\
QX_{k-1} & \longrightarrow & X_{k-1} & \longrightarrow & X_k \\
\downarrow & & \downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle q_{k-1}} \\
S^{2nk-1} & \xrightarrow{s_{k-1}} & J_{k-1}(S^{2n}) & \longrightarrow & J_k(S^{2n})
\end{array}
$$

By Lemma 4.1 there is a decomposition $QX_{k-1} \simeq S^{2nk-1} \times Y$ and a homotopy pushout

$$
\begin{array}{ccc}
QX_{k-1} \simeq S^{2nk-1} \times Y & \xrightarrow{\theta_{k-1}} & X_{k-1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow \\
Y & \xrightarrow{\delta_k} & X_k
\end{array}
$$

for some map θ_{k-1}. By hypothesis, there is a map $X_{k-1} \xrightarrow{\varepsilon_{k-1}} W_{np}$. Now we check that the two hypotheses of Theorem 4.3 are satisfied. First, by Lemma 7.3, the map $S^{2nk-1} \times Y \xrightarrow{\theta_{k-1}} X_{k-1}$ in this pushout can be taken to be the composite $S^{2nk-1} \times Y \xrightarrow{p_{k-1} \times 1} P_{k-1} \times Y \longrightarrow X_{k-1}$, where $\Sigma p_{k-1} \simeq \overline{t}_{k-1} \circ \underline{p}^r$ for some map $\overline{p}_{k-1}$. Second, as W_{np} is a loop space it is homotopy associative, and by Lemma 3.1 its pth-power map is null homotopic. Thus Theorem 4.3 implies that there is an extension

$$
\begin{array}{ccc}
X_{k-1} & \xrightarrow{\varepsilon_{k-1}} & W_{np} \\
\downarrow & & \| \\
X_k & \xrightarrow{\varepsilon_k} & W_{np}
\end{array}
$$

for some map ε_k. $\square$

8. Consequences

In this section we explore some consequences of our proof of Theorem 1.1 in order to round off the picture. First, in Lemma 3.1 we used a factorization of the pth-power map on $\Omega^2 S^{2np+1}$ through the double suspension to show that the pth-power map on BW_{np} is null homotopic. Now that we know the pth-power map on $\Omega^2 S^{2n+1}$ factors through the double suspension we can use the same argument to improve from the np-case to all cases.

Theorem 8.1. *The pth-power map on BW_n is null homotopic.* $\square$
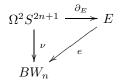
Now return to the construction of the extension

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_E} & E \\
\downarrow{\scriptstyle S} & \swarrow{\scriptstyle e_1} & \\
\Omega^2 S^{2np+1}\{p\} & &
\end{array}
$$

in Proposition 2.1. The only properties of $\Omega^2 S^{2np+1}\{p\}$ and S used in the proof were: (i) S is an H-map, (ii) $\Omega^2 S^{2np+1}\{p\}$ is a homotopy associative H-space, and (iii) the pth-power map on $\Omega^2 S^{2np+1}\{p\}$ is null homotopic. Recall that the map $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ is an H-map. By [G] for $p \geq 5$ and [T2] for $p = 3$, BW_n is a homotopy associative H-space. Thus, now that we know the pth-power map on BW_n is null homotopic we can replace $\Omega^2 S^{2np+1}\{p\}$ by BW_n in the proof of Proposition 2.1 to obtain the following.

Proposition 8.2. *There is an extension*

$$
\begin{array}{ccc}
\Omega^2 S^{2n+1} & \xrightarrow{\partial_E} & E \\
\downarrow{\scriptstyle \nu} & \swarrow{\scriptstyle e} & \\
BW_n & &
\end{array}
$$

for some map e. $\square$

Proposition 8.2 recovers the key result in [GT]. It allows for a reformulation of $T^{2n+1}(p^r)$ which satisfies additional properties. Since ∂_E factors as the composite $\Omega^2 S^{2n+1} \longrightarrow \Omega S^{2n+1}\{p^r\} \longrightarrow E$, there is a homotopy pullback diagram

$$
\begin{array}{ccc}
S^{2n-1} \longrightarrow T^{2n+1}(p^r) \longrightarrow \Omega S^{2n+1} \\
\downarrow{\scriptstyle E^2} \qquad\qquad \downarrow \qquad\qquad \| \\
\Omega^2 S^{2n+1} \longrightarrow \Omega S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1} \\
\downarrow{\scriptstyle \nu} \qquad\qquad \downarrow \qquad\qquad \\
BW_n =\!=\!=\!=\!= BW_n
\end{array}
$$

which reformulates the construction of $T^{2n+1}(p^r)$. Further, there is a homotopy pullback diagram

$$
\begin{array}{ccc}
\Omega P^{2n+1}(p^r) \longrightarrow T^{2n+1}(p^r) \longrightarrow X \\
\| \qquad\qquad\qquad \downarrow \qquad\qquad \downarrow \\
\Omega P^{2n+1}(p^r) \xrightarrow{\Omega i} \Omega S^{2n+1}\{p^r\} \longrightarrow E \\
\downarrow \qquad\qquad\qquad \downarrow \\
BW_n =\!=\!=\!=\!= BW_n
\end{array}
$$

which implies the additional property that Ωi factors through $T^{2n+1}(p^r)$. Carrying on, a key property of [CMN1] regarding the homotopy theory of the Moore space can be recovered, namely, that the inclusion $\iota\colon S^{2n-1} \longrightarrow \Omega F$ of the bottom cell has a left homotopy inverse. Since Ωi factors through

$T^{2n+1}(p^r)$, so does Ωq. Thus there is a homotopy pullback diagram

$$
\begin{array}{ccccc}
\Omega F & \longrightarrow & \Omega P^{2n+1}(p^r) & \xrightarrow{\ \Omega q\ } & \Omega S^{2n+1} \\
\downarrow{\scriptstyle r} & & \downarrow & & \| \\
S^{2n-1} & \longrightarrow & T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1}
\end{array}
$$

which defines the map r. By connectivity, the entire left square is degree one in $H_{2n-1}(\ )$, and so $r \circ \iota$ is degree one, implying that it is homotopic to the identity map.

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